Tutorial 10
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BACK

Questions of today

1. Let $g : \Omega' \to \mathbb{C}$ be holomorphic, and $f : \Omega \to \Omega'$ be harmonic. Show that $f \circ g$ is harmonic.
2. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let Ω be a domain of $\mathbb{C}$. Let $f : \mathbb{D} \to \Omega$ be a confomal map with power series expansion at 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Show that the area of Ω is given by $\pi \sum_{n=0}^{\infty} n|a_n|^2$.
3. Let $f : \Omega \to \Omega'$ be conformal, suppose f can be extended continuously at some point $a \in \partial\Omega$, show that $f(a) \notin \Omega'$.
4. Show that the punctured disc $\mathbb{D} \setminus \{0\}$ is not conformally equivalent to the annulus $\{z \in \mathbb{C} : r < |z| < R\}$, where $R > r > 0$.
5. Let $\Omega \neq \mathbb{C}$ be a simply connected domain, and $f : \Omega \to \Omega$ be holomorphic with at least two fixed points. Show that f is the identity.
6. Let $\Omega \neq \mathbb{C}$ be a simply connected domain, and a, b be two points in Ω. Find all the automorphism (comformal mapping onto itself) $\Omega \setminus \{a, b\} \to \Omega \setminus \{a, b\}$.

Hints & solutions of today

1. Use multivariable chain rule. One useful characterization of harmonic function is : f harmonic iff $f_{z\bar{z}} = 0$.
2. Use change of variable formula.
3. If $f(a) = f(z)$ for some z in Ω. Choose disjoint neighberhoods V and U of a and z. $f(U)$ contains a delta neighbourhood of $f(z)$, while $f(U) \cap f(V \cap \Omega) = \emptyset$ by the injectivity of f. Use this to derive a contradiction.
4. By Riemann's extension theorem, such a conformal map would induce a map from $\mathbb{D}$ to the closure of the annulus. Such an extension must be injective by question 3, and hence must be a conformal map to its image(Injective implies conformal). Finally, use Cauchy theorem to show that the map cannot be conformal.
5. Reduce the question to the case $\Omega = \mathbb{D}$. Then apply Schwarz lemma.
6. Reduce the question to the case $\Omega = \mathbb{D}$, and show that the conformal map extends to a conformal map $\mathbb{D} \to \mathbb{D}$ sending $\{a, b\}$ to $\{a, b\}$. (You need to use question 3, but why can't a and b be sent to the boundary of $\mathbb{D}$?)
